An Exploration of Stable Cycles in Dynamical Systems

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Abstract

Many physical systems consist of matter moving smoothly in some fixed space, such as a ball rolling down a hill or flows of fluid or electricity. These situations can be modeled by abstract mathematical models called dynamical systems.

The theory of continuous dynamical systems is an extension of the theory of ordinary differential equations. In this theory, one considers the time-dependent behavior of points in a manifold when their velocity is given by a vector field over the space. Depending on the vector field and location of the points, they can converge to a fixed point, diverge, undergo semi-convergence, or fall into a cycle where they return to the same position after a fixed time.

In this research, I study the conditions required for stable cycles in continuous dynamical systems in $\mathbb{R}^n$, in particular the existence of cycles of varying period, and how the system can be altered to preserve the cycles. Using online vector field simulators and numerical programming languages such as Julia, I find approximate numerical solutions to the differential equations to test my hypothesis that the system stays stable after even functions of x and y are added to the x- and y- components of the vector fields, respectively.

Dynamical systems and their behavior under continuous deformations are the basis of chaos theory and much of mathematical physics. Furthermore, dynamical systems have applications to circuits and image processing.

1 Introduction

Consider the theory of ordinary differential equations. In this theory, equations of the form $y'(t) = f(y(t))$, where f is a function of y only, are called autonomous differential equations. Tracing the graphs of solutions to an autonomous equation in the ty-plane, we might find some constant c where $y(t) = c$ is a solution. It is easy to see that such constants are precisely the zeroes of the function f. They are called *fixed points* or *equilibria* of the system governed by the equation. Varying the initial condition $y(0)$ may produce convergence toward or divergence away from the fixed points. Furthermore, continuously varying the defining function f may affect the behavior of solutions around these points. This paper looks at the theory of fixed points of dynamical systems, which is a generalization of these notions to multivariable systems of equations in higher dimensions.

2 Dynamical systems overview

A *dynamical system* is a manifold M endowed with a set of functions $\Phi^t : M \to M$ for all $t \in T$, the time dimension. Each function describes the state of the system after a certain time t. In this paper, I will be looking exclusively at the case where $M = \mathbb{R}^n$, $T = \mathbb{R}^+$, and Φ^t is defined by an autonomous differential equation $\mathbf{x}' = \mathbf{f}(\mathbf{x})$, where $\mathbf{f} : M \to M$ is a smooth function along all input dimensions and $\mathbf{x}$ is an unknown function mapping $T \to M$.

First assume that f is continuously differentiable and $\mathbf{f}(\mathbf{c})$ is defined for all $\mathbf{c} \in M$. By the Peano-Cauchy theorem, the initial value problem

$$\mathbf{x}' = \mathbf{f}(\mathbf{x}), \mathbf{x}(0) = \mathbf{c} \tag{1}$$

has a solution for all $\mathbf{c} \in M$, and by the Picard-Lindelof theorem the solution is unique, because continuous differentiability implies that f is locally Lipschitzian [1]. Call each of these solutions $x_c(t)$. For all $t \in T$ and $c \in M$, define $\Phi^t(c) = \mathbf{x}_c(t)$. Note that if f is plotted as a vector field, then Φ^t describes the location of each point after following the vector field for a period of time t.

2.1 Fixed points

Fixed points are points $\mathbf{c} \in M$ such that $\mathbf{f}(\mathbf{c}) = \mathbf{0}$. Whenever $\mathbf{c}$ is a fixed point, we have $\mathbf{x}_\mathbf{c}(t) = \mathbf{c}$ and hence $\Phi^t(\mathbf{c}) = \mathbf{c}$ for all $t \in T$.

3 Linear dynamical systems

First we consider the case where $\mathbf{f}(\mathbf{c}) = A\mathbf{c}$, where A is an n by n matrix. The point $\mathbf{c} = \mathbf{0}$ is a fixed point. Assume A is invertible; then $\mathbf{c} = 0$ is the only fixed point. How does Φ^t behave for $\mathbf{c}$ near $\mathbf{0}$?

We first show that the set S of functions $\mathbf{x}_\mathbf{c}$ is a subspace of the vector space of all functions $\mathbf{x} : t \to M$ by showing that it is closed under addition and scalar multiplication of functions. First, let $\mathbf{c}_1, \mathbf{c}_2 \in M$. Then $\mathbf{x}'_{\mathbf{c}_1}(t) = A\mathbf{x}_{\mathbf{c}_1}(t)$ and $\mathbf{x}'_{\mathbf{c}_2}(t) = A\mathbf{x}_{\mathbf{c}_2}(t)$. Let $\mathbf{y}(t) = \mathbf{x}_{\mathbf{c}_1}(t) + \mathbf{x}_{\mathbf{c}_2}(t)$. Then $\mathbf{y}'(t) = \mathbf{x}'_{\mathbf{c}_1}(t) + \mathbf{x}'_{\mathbf{c}_2}(t) = A\mathbf{x}_{\mathbf{c}_1}(t) + A\mathbf{x}_{\mathbf{c}_2}(t) = A(\mathbf{x}_{\mathbf{c}_1}(t) + \mathbf{x}_{\mathbf{c}_2}(t)) = A\mathbf{y}(t)$

and from the initial condition $\mathbf{y}(0) = \mathbf{c}_1 + \mathbf{c}_2$, it follows that $\mathbf{y}(0) = \mathbf{x}_{\mathbf{c}_1+\mathbf{c}_2}(t)$ and thus $\mathbf{y} \in S$. Next, let $\mathbf{c} \in M$ and $a \in \mathbb{R}$. Then $\mathbf{x}'_{\mathbf{c}}(t) = A\mathbf{x}_{\mathbf{c}}(t)$. Let $\mathbf{z}(t) = a\mathbf{x}_{\mathbf{c}}(t)$. Then $\mathbf{z}'(t) = a\mathbf{x}'_{\mathbf{c}}(t) = a \cdot (A\mathbf{x}_{\mathbf{c}}(t)) = A(a\mathbf{x}_{\mathbf{c}}(t)) = A\mathbf{z}(t)$. It follows that $\mathbf{z}(t) = \mathbf{x}_{a\mathbf{c}}(t)$ and thus $\mathbf{z} \in S$. Finally, notice that the function $\mathbf{x}_{\mathbf{0}}(t) = \mathbf{0}$ is the zero vector of S. It follows that S is a vector space.

Consider the points $\mathbf{c}$ such that $\mathbf{f}(\mathbf{c})$ is parallel to $\mathbf{c}$. These are the eigenvectors of A. When $\mathbf{c}$ is an eigenvector corresponding to a real eigenvalue λ, we have $\mathbf{x}'_{\mathbf{c}}(t) = \lambda\mathbf{x}_{\mathbf{c}}(t)$. In fact, because $\mathbf{f}$ is a linear transformation, for any scalar multiple $\mathbf{d}$ of $\mathbf{c}$ it holds that $\mathbf{x}'_{\mathbf{d}}(t) = \lambda\mathbf{x}_{\mathbf{d}}(t)$. Solving the differential equation we obtain $\mathbf{x}_{\mathbf{c}}(t) = e^{\lambda t}\mathbf{c}$. We see quantitatively that $\mathbf{x}_{\mathbf{c}}(t)$ converges if and only if λ is negative.

Now consider $\mathbf{c} \in M$ such that $\mathbf{c} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3 + ... + a_m\mathbf{v}_m$, where $m \leq n$ and $\mathbf{v}_1$ through $\mathbf{v}_m$ are eigenvectors of A corresponding to the real distinct eigenvalues λ_1 to λ_m. Because S is closed under linear combinations, we see that $\mathbf{x}_{\mathbf{c}} \in S$ and $\mathbf{x}_{\mathbf{c}}(t) = a_1\mathbf{x}_{\mathbf{v}_1}(t) + a_2\mathbf{x}_{\mathbf{v}_2}(t) + ... + a_m\mathbf{x}_{\mathbf{v}_m}(t) = a_1e^{\lambda_1 t}\mathbf{v}_1 + a_2e^{\lambda_2 t}\mathbf{v}_2 + ... + a_me^{\lambda_m t}\mathbf{v}_m$. If all the eigenvalues are real, $\mathbf{c}$ can range over M with appropriate choices of scalars $a_1, a_2...a_m$ and vectors $\mathbf{v}_1, \mathbf{v}_2...\mathbf{v}_m$ from each eigenspace. From the above expression of $\mathbf{x}_{\mathbf{c}}(t)$, we can see that if all eigenvalues are real negative numbers, then $\mathbf{x}_{\mathbf{c}}(t)$ converges to 0 as $t \to \infty$ for all $\mathbf{c}$, but if some eigenvalue λ is a non-negative real, then $\mathbf{x}_{\mathbf{v}}(t)$ does not converge where $\mathbf{v}$ is an eigenvector corresponding to λ. The cases with complex eigenvalues are similar. [2]

Solutions of the system $\mathbf{x}' = A\mathbf{x}$ may take different forms, depending on the characteristics of the eigenvalues.

Case 1: Real unique eigenvalues

For each real unique eigenvalue λ with associated eigenvector $\mathbf{v}$, $\mathbf{x} = e^{\lambda t}\mathbf{v}$ is a solution. These correspond to trajectories that are stationary ($\lambda = 0$) or have exponential growth or decay in straight lines.

Case 2: Complex eigenvalues

If $\lambda_{re} \pm i\lambda_{im}$ is a pair of complex conjugate eigenvalues with eigenvectors $\mathbf{v}_{re} \pm i\mathbf{v}_{im}$, then there are linearly independent solutions of the form $\mathbf{x}_1 = e^{\lambda_{re}t}(\mathbf{v}_{re}\cos(\lambda_{im}t) - \mathbf{v}_{im}\sin(\lambda_{im}t))$ and $\mathbf{x}_2 = e^{\lambda_{re}t}(\mathbf{v}_{re}\sin(\lambda_{im}t) + \mathbf{v}_{im}\cos(\lambda_{im}t))$ corresponding to initial values $\mathbf{v}_{re}$ and $\mathbf{v}_{im}$ respectively. These correspond to spiral-shaped trajectories, which either spiral inward to the origin, spiral outward exponentially, or produce a stable elliptical path around the origin. The initial point of the trajectory can be any linear combination of $\mathbf{v}_{re}$ and $\mathbf{v}_{im}$.

Case 3: Real repeated eigenvalues with the same geometric multiplicities

If λ is an eigenvalue with multiplicity m associated with an m-dimensional eigenspace with basis vectors $\mathbf{v}_1, \mathbf{v}_2...\mathbf{v}_m$, then $\mathbf{x}_1 = e^{\lambda t}\mathbf{v}_1$, $\mathbf{x}_2 = e^{\lambda t}\mathbf{v}_2$, ... $\mathbf{x}_m = e^{\lambda t}\mathbf{v}_m$ are linearly independent solutions. As in case 1, they correspond to trajectories that are stationary or have exponential growth and decay in straight lines.

Case 4: Real repeated eigenvalues with smaller geometric multiplicities

If λ is an eigenvalue with multiplicity m associated with a k-dimensional eigenspace, where $k < m$, with basis vectors $\mathbf{v}_1, \mathbf{v}_2 ... \mathbf{v}_k$, then k of the solutions are given by $\mathbf{x}_1 = e^{\lambda t}\mathbf{v}_1$, $\mathbf{x}_2 = e^{\lambda t}\mathbf{v}_2$ ... $\mathbf{x}_k = e^{\lambda t}\mathbf{v}_k$. The other solutions are found by choosing $m - k$ vectors $\mathbf{u}_1, \mathbf{u}_2 ... \mathbf{u}_{m-k}$ to satisfy the equations:
$(A - \lambda I)\mathbf{u}_1 = \mathbf{v}$
$(A - \lambda I)\mathbf{u}_2 = \mathbf{u}_1$
$(A - \lambda I)\mathbf{u}_3 = \mathbf{u}_2$
...
$(A - \lambda I)\mathbf{u}_{m-k} = \mathbf{u}_{m-k-1}$
The vectors u_1, u_2, ..., u_{m-k-1} are called **generalized eigenvectors**. In this case, $\mathbf{v}$ can be any linear combination of $\mathbf{v}_1, \mathbf{v}_2 ... \mathbf{v}_k$ such that the equations have solutions.
The solutions have exponential growth and decay along a path which is nearly linear but not quite. If $\lambda = 0$, the solutions are not stable but instead can be expressed as a polynomial of t with vector coefficients.
Case 5: Complex repeated eigenvalues with the same geometric multiplicity
If $\lambda_{re} \pm i\lambda_{im}$ is a conjugate pair of eigenvalues with multiplicity m, associated with a $2m$-dimensional eigenspace with basis vectors $\mathbf{v}_{1,re} \pm i\mathbf{v}_{1,im}, \mathbf{v}_{2,re} \pm i\mathbf{v}_{2,im} ... \mathbf{v}_{m,re} \pm i\mathbf{v}_{m,im}$, then $\mathbf{x}_1 = e^{\lambda_{re} t}(\mathbf{v}_{1,re}\cos(\lambda_{im}t) - \mathbf{v}_{1,im}\sin(\lambda_{im}t))$, $\mathbf{x}_2 = e^{\lambda_{re} t}(\mathbf{v}_{2,re}\cos(\lambda_{im}t) - \mathbf{v}_{2,im}\sin(\lambda_{im}t))$ ... $\mathbf{x}_m = e^{\lambda_{re} t}(\mathbf{v}_{m,re}\cos(\lambda_{im}t) - \mathbf{v}_{m,im}\sin(\lambda_{im}t))$ and $\mathbf{x}_{m+1} = e^{\lambda_{re} t}(\mathbf{v}_{1,re}\sin(\lambda_{im}t) + \mathbf{v}_{1,im}\cos(\lambda_{im}t))$, $\mathbf{x}_{m+2} = e^{\lambda_{re} t}(\mathbf{v}_{2,re}\sin(\lambda_{im}t) + \mathbf{v}_{2,im}\cos(\lambda_{im}t))$ ... $\mathbf{x}_{2m} = e^{\lambda_{re} t}(\mathbf{v}_{m,re}\sin(\lambda_{im}t) + \mathbf{v}_{m,im}\cos(\lambda_{im}t))$. These solutions correspond to initial values of $\mathbf{v}_{1,re}, \mathbf{v}_{2,re} ... \mathbf{v}_{m,re}, \mathbf{v}_{1,im}, \mathbf{v}_{2,im} ... \mathbf{v}_{m,im}$ respectively. The solutions have spiraling behavior that is similar to case 2.
Case 6: Complex repeated eigenvalues with smaller geometric multiplicities
As in case 4, if $\mathbf{v}_1 = \mathbf{v}_{1,re} \pm i\mathbf{v}_{1,im}, ... \mathbf{v}_k = \mathbf{v}_{k,re} \pm i\mathbf{v}_{k,im}$ are eigenvectors for the complex eigenvalue $\lambda_{re} + i\lambda_{im}$ and form a basis for an eigenspace of dimension $k < m$, then k of the solutions are given by $\mathbf{x}_1 = e^{\lambda t}\mathbf{v}_1$, $\mathbf{x}_2 = e^{\lambda t}\mathbf{v}_2$ ... $\mathbf{x}_k = e^{\lambda t}\mathbf{v}_k$. As before, the generalized eigenvectors $u_1, \ldots, u_{m-k}$ are found by solving:
$(A - \lambda I)\mathbf{u}_1 = \mathbf{v}$
$(A - \lambda I)\mathbf{u}_2 = \mathbf{u}_1$
$(A - \lambda I)\mathbf{u}_3 = \mathbf{u}_2$
...
$(A - \lambda I)\mathbf{u}_{m-k} = \mathbf{u}_{m-k-1}$
The solutions have similar spiraling properties to case 5, depending on whether λ_{re} is positive, negative, or 0.

3.1 Phase Changes in Linear Systems

In a family of dynamical systems $\mathbf{x}' = f(\mathbf{x}, \mu)$, a **phase change** occurs when a stable equilibrium point becomes semistable or unstable under continuous variation of f. Here, we describe the dynamical system as changing with variable parameter μ.
When the system is linear, we have $\mathbf{x}' = A[\mu]x$, where A is a matrix as a func-

tion of μ. When the entries of A vary continuously, several types of changes to the roots of characteristic polynomial can occur, causing changes to the nature of the fixed point $\mathbf{x}_0 = \mathbf{0}$. We denote the polynomial by $p[\mu](\lambda)$ and its roots by λ_0, λ_1, etc.

1. $\lambda_0 \in \mathbb{R}$: $\lambda_0 < 0 \leftrightarrow \lambda_0 > 0$. In this case, $\mathbf{x}_0 = \mathbf{0}$ may transition from a stable point to a semistable point, or from a semistable point to an unstable one, or vice versa. Scalar multiples of $\mathbf{v}_0$, where $\mathbf{v}_0$ is the eigenvector associated with λ_0, go from being stable initial conditions to unstable ones.

2. $\lambda_0 ... \lambda_k \in \mathbb{R}$: $\lambda_0 ... \lambda_k < 0 \leftrightarrow \lambda_0 ... \lambda_k > 0$. Multiple roots can pass through zero at the same time. When this happens, a multidimensional space of inputs goes from stable to unstable.

3. $\lambda_0, \lambda_1 \in \mathbb{C}$: $Im(\lambda_0) \neq 0; \lambda_1 = \bar{\lambda_0}; Re(\lambda_0) < 0 \leftrightarrow Im(\lambda_0) = Im(\lambda_1) = 0; \lambda_0, \lambda_1 < 0$. The origin stays a stable point for the appropriate space of initial conditions, but solutions change from having spiral trajectories to straight ones.

4. $\lambda_0, \lambda_1 \in \mathbb{C}$: $Im(\lambda_0) \neq 0; \lambda_1 = \bar{\lambda_0}; Re(\lambda_0) < 0 \leftrightarrow Im(\lambda_0) = Im(\lambda_1) = 0; \lambda_0 < 0; \lambda_1 > 0$. The same as #3 but spiral solutions become semistable solutions instead of stable ones.

5. $\lambda_0, \lambda_1 \in \mathbb{C}$: $Im(\lambda_0) \neq 0; \lambda_1 = \bar{\lambda_0}; Re(\lambda_0) > 0 \leftrightarrow Im(\lambda_0) = Im(\lambda_1) = 0; \lambda_0, \lambda_1 > 0$. The origin is an unstable point for the appropriate space of initial conditions, but solutions change from having spiral trajectories to straight ones.

6. $\lambda_0, \lambda_1 \in \mathbb{C}$: $Im(\lambda_0) \neq 0; \lambda_1 = \bar{\lambda_0}; Re(\lambda_0) > 0 \leftrightarrow Im(\lambda_0) = Im(\lambda_1) = 0; \lambda_0 < 0, \lambda_1 > 0$. The same as #5 but spiral solutions become semistable solutions instead of unstable ones.

7. $\lambda_0, \lambda_1 \in \mathbb{C}$: $Im(\lambda_0) > 0; Im(\lambda_1) = -Im(\lambda_0); Re(\lambda_0) = Re(\lambda_1) = 0 \leftrightarrow Im(\lambda_0) = Im(\lambda_1) = 0; \lambda_0 > 0; \lambda_1 < 0$. Solutions in the 2D space of appropriate initial conditions change from having metastable orbitals around the origin to semistable trajectories.

If $M = \mathbb{R}^2$, the matrix $A[\mu]$ describing the system can be written as:

$$A[\mu] = \begin{bmatrix} f_{1,1}(\mu) & f_{1,2}(\mu) \\ f_{2,1}(\mu) & f_{2,2}(\mu) \end{bmatrix}$$

Then, the roots of the equation can be described by two parameters:

$$p(\mu) = -\frac{f_{1,1}(\mu) + f_{2,2}(\mu)}{2}$$

$$q(\mu) = \frac{f_{1,1}(\mu)^2 - 2f_{1,1}(\mu)f_{2,2}(\mu) + 2f_{1,2}(\mu)f_{2,1}(\mu) + f_{2,2}(\mu)^2}{4}$$

The possible states of the system listed above correspond to various conditions on p and q. If $q(\mu) < 0$, then the eigenvalues are complex with real part equal to $p(\mu)$. If $q(\mu) = 0$, then the eigenvalues are both equal to $p(\mu)$. If $q(\mu) > 0$, then the eigenvalues are real and positive if $q(\mu) < p(\mu)^2$ and $p(\mu) > 0$, real and negative if $q(\mu) < p(\mu)^2$ and $p(\mu) < 0$, or real with opposite signs if $q(\mu) > p(\mu)^2$.

4 Extension to nonlinear dynamical systems

In a continuous dynamical system defined with a nonlinear function of $\mathbf{x}$, the rectification theorem guarantees that if $\mathbf{f}(\mathbf{x}_0) \neq 0$, then the streamlines in the neighborhood of $\mathbf{x}_0$ can be converted into parallel lines using a smooth change of coordinates. Similarly, where $\mathbf{f}(\mathbf{x}_0) = \mathbf{0}$, the vectors in the neighborhood of $\mathbf{x}_0$ can be approximated by $\mathbf{x}' = J(\mathbf{x} - \mathbf{x}_0)$, where $J = \frac{\partial \mathbf{f}}{\partial \mathbf{x}}$ is the Jacobian matrix of $\mathbf{f}$ at $\mathbf{x} = \mathbf{x}_0$.

The properties of the corresponding linear system $\mathbf{x}' = J\mathbf{x}$ are similar to those of the system $\mathbf{x}' = \mathbf{f}(\mathbf{x})$. If $\mathbf{x} = \mathbf{0}$ is a stable equilibrium point in the linear system (that is, if all eigenvalues of J have negative real part), then $\mathbf{x} = \mathbf{x}_0$ is a stable equilibrium point in the non-linear system with eigenvector $\mathbf{x} = \mathbf{v}$ corresponding to the trajectory of $\mathbf{x} = \mathbf{x}_0 + \mathbf{v}$ in the non-linear system. Similarly, if an eigenvalue λ of J has positive real part and corresponding eigenvector $\mathbf{v}$, then $\mathbf{x} = \mathbf{x}_0 + \mathbf{v}$ will be an unstable point in the non-linear system. However, if an eigenvalue λ is 0 or imaginary, indicating that the eigenvectors $\mathbf{v}$ are fixed points or form stable orbits in the linear system respectively, points $\mathbf{x}_0 + \mathbf{v}$ may not necessarily undergo the same behavior in the non-linear system. For example, the system defined by

$$x_1' = x_1^3 - x_2$$

$$x_2' = x_1 + x_2^3$$

has Jacobian

$$J(\mathbf{x}) = \begin{bmatrix} 3x_1^2 & -1 \\ 1 & 3x_2^2 \end{bmatrix}$$

and fixed point $\mathbf{x}_0 = \mathbf{0}$, giving Jacobian

$$J = J(\mathbf{0}) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

. This matrix has eigenvalues $\lambda_0, \lambda_1 = \pm i$, indicating that the solutions to the linear system orbit the origin counterclockwise. However, the solutions $\mathbf{x_c}(t)$ to the non-linear system spiral farther and farther outward for any value $\mathbf{c} \neq \mathbf{0}$, and thus, with the exception of $\mathbf{x_0}(t)$, every solution is unstable.

For other systems, such as the system defined by $\mathbf{x}' = \begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} f_1(x) \\ f_2(x) \end{bmatrix} = \begin{bmatrix} x_1^2 - x_2 \\ x_1 + x_2^2 \end{bmatrix}$, there are still periodic orbits, as suggested by the Jacobian matrix, but with different period varying with distance from the origin.

In the above case, the orbits can be proved to be periodic by considering $\mathbf{x}'$ as a vector field on $\mathbb{R}^2$ and noting the symmetry of $\mathbf{x}'$ with respect to the line $x_1 = x_2$. But in general there is no obvious way to tell whether the orbits are periodic.

In my research, I used a computer program to numerically solve the ODEs of similar systems to determine whether or not they formed cycles. Focusing on

systems of the form $\mathbf{x}' = \begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} f(x_1) - x_2 \\ x_1 + g(x_2) \end{bmatrix}$, I created a program in the Julia programming language that used a 2nd order Taylor method to plot the trajectories of points close to the origin. I used numerical differentiation (or sometimes pre-programmed functions) to calculate the derivatives of f and g.
First, I conjectured that whenever $f(x)$ and $g(x)$ were of the form ax^2, where a was a positive constant that was different for f and g, then all cycles in a neighborhood of the origin were perfectly stable; they returned to their exact initial position at some time $t > 0$. I then extended my conjecture to include the case where f and g were even polynomials of degree greater than 2. Based on low-accuracy approximations this conjecture seemed to hold, and in fact when $f = g$ I could prove that it held. However, unlike in the linear case, the apparent period of the cycles varied with distance from the origin.
However, upon applying higher approximation, the solutions did not converge noticeably closer to closed loops. For example, I considered the system $\mathbf{x}' = \begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} f(x_1) - x_2 \\ x_1 + g(x_2) \end{bmatrix}$ with $f(x) = 0.7x^2$, $g(x) = x^2$ with initial point $\mathbf{x}(0) = \begin{bmatrix} 0.3 \\ 0 \end{bmatrix}$. The resulting trajectory crossed the positive x-axis for a second time at $t = 0.662$. Although a numerical approximation with $\Delta t = 0.01$ yielded $||\mathbf{x}(0.662) - \mathbf{x}(0)|| < 0.0025$, decreasing the step size to $\Delta t = 0.001$ did not yield a noticeable decrease in the value $||\mathbf{x}(0.662) - \mathbf{x}(0)||$. As the Taylor method was expected to converge to the exact solution upon increasing precision, I concluded that the "cycles" were likely not actually closed loops but rather spiraled outward by about 0.002 each period.

4.1 Further research

Although conjectures and numerical approximations do not prove anything, they stimulate further research into dynamical systems. Perhaps somewhere there is a theorem that answers the questions I address in this paper.
For higher-order dynamical systems, fixed points can be analyzed using normal form. The **normal form** of a dynamical system is another dynamical system described by a simpler equation that preserves the most important properties of the original system. If a system can be described by the equation $\mathbf{x}' = A\mathbf{x} + \mathbf{a}_1(\mathbf{x}) + \mathbf{a}_2(\mathbf{x}) + \ldots$, where A is an $n \times n$ matrix and $a_j(x)$ is an n-vector of homogeneous polynomials of degree $j+1$ whose variables are the entries of $\mathbf{x}$, then it is possible to introduce a change of variables $\mathbf{y}' = \mathbf{u}_1(y) + u_2(y) + \ldots$ such that the new expression $\mathbf{y}$ is in some way simpler than the old one [4].
In four dimensions and above, stable cycles can involve two mutually perpendicular planes with independent elliptical trajectories, and their non-linear approximations. In linear systems, if the eigenvalues have no least common multiple, the point may never return to its initial position yet it may still be confined to a submanifold given by the Cartesian product of two ellipses.

5 References

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